Mean Curvature Regularization

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This review is based on the paper 'Augmented Lagrangian Method for A Mean Curvature Based Image Denoising Model' by Wei Zhu, Xue-Cheng Tai and Tony Chan (2013).

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Outline

Motivation

- A question rose up in *classical* differential geometry: given a 2D continuous closed curve with no self-intersecting (called Jordan curve), what's the minimal surface bounded by this curve?
  In other words, find a minimal surface with a boundary constrain.
- Relation to imaging: **Denoising**.

Calculus Review

- Given a surface S $u(x, y) = z$, the area of S, denoted as A_u, in domain Ω can be calculated by

$$A_u = \int_\Omega \sqrt{1 + u_x^2 + u_y^2} = \int_\Omega \sqrt{1 + |\nabla u|^2}$$

  Perturb it by a small proportion of other surface we get $u + t\eta$, $t \in \mathbb{R}$ and η is a smooth surface with $\eta|_{\partial\Omega} = 0$. Then

$$A_{(u+t\eta)} = \int_\Omega \sqrt{1 + |\nabla u + t\nabla\eta|^2}$$

  Then

$$\frac{d}{dt}_{t=0} A_{(u+t\eta)} = \int_\Omega \frac{\langle \nabla u, \nabla\eta \rangle}{\sqrt{1 + |\nabla u|^2}} = -\int_\Omega \eta \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)$$

Differential Geometry

- Let $k = \nabla \cdot (\frac{\nabla u}{\sqrt{1+|\nabla u|^2}})$, if $k = 0$ then u is a critical/saddle point. In fact, such u has a minimal surface over all. Here k is called **mean curvature**. Same as the one in differential geometry derived by the first fundamental form, where k is defined as $k = \frac{k_1+k_2}{2}$, k_1 and k_2 are the two principle curvatures, maximum and minimum curvature correspondingly, and mathematically, they are the eigenvalues of *Weigarten map*.
- Now we have the mean curvature (MC) denoising model:

$$\min \lambda \int |\nabla \cdot (\frac{\nabla u}{\sqrt{1+|\nabla u|^2}})| + \frac{1}{2} \int (f-u)^2$$

Given $(x, y, f(x, y))$ as noisy image surface.

Details

- Write everything out:

$$k = \frac{(1+u_x^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_y^2)u_{yy}}{2(1+u_x^2+u_y^2)^{3/2}}$$

- MC model:

$$\min \lambda \int |k| + \frac{1}{2}\int (f-u)^2$$

- One good try, adding relaxation (kernel method):

$$\min \lambda \int |\Phi(k)| + \frac{1}{2}\int (f-u)^2$$

With

$$\Phi(x) = \begin{cases} x^2 & \text{if } |x| \leq 1 \\ |x| & \text{if } |x| > 1. \end{cases}$$

However, I believe this is a typo, as this idea comes from Huber loss:

$$H(x) = \begin{cases} x^2/2 & \text{if } |x| \leq \delta \\ \delta(|x| - \delta/2) & \text{if } |x| > \delta. \end{cases}$$

Outline

Problem Rewriting Step by Step

Original problem:

$$\min \lambda \int_\Omega |\nabla \cdot (\frac{\nabla u}{\sqrt{1+|\nabla u|^2}})| + \frac{1}{2}\int (f-u)^2$$

$\Rightarrow$ (relax k)

$$\min_{u,q,\mathbf{w},\mathbf{p}} \{\lambda \int_\Omega |q| + \frac{1}{2}\int (f-u)^2\}$$

$$\textit{with } q = \nabla \cdot \mathbf{n},\ \mathbf{n} = \frac{\mathbf{p}}{\sqrt{1+|\mathbf{p}|^2}},\ \mathbf{p} = \nabla u.$$

$$\begin{aligned} E(u,q,\mathbf{n},\lambda_1,\boldsymbol{\lambda_2},\boldsymbol{\lambda_3}) = {} & \lambda \int_\Omega |q| + \frac{1}{2}\int (f-u)^2 + \frac{r_1}{2}\int (q - \nabla \cdot \mathbf{n})^2 \\ & + \int \lambda_1 (q - \nabla \cdot \mathbf{n}) + \frac{r_2}{2}\int (n - \frac{\mathbf{p}}{\sqrt{1+|\mathbf{p}|^2}})^2 \\ & + \int \boldsymbol{\lambda_2} \cdot (\mathbf{n} - \frac{\mathbf{p}}{\sqrt{1+|\mathbf{p}|^2}}) + \frac{r_3}{2}\int (\mathbf{p} - \nabla u)^2 \\ & + \int \boldsymbol{\lambda_3} \cdot (\mathbf{p} - \nabla u) \end{aligned}$$

Problem Rewriting Step by Step

(cont'd) $\Rightarrow$

$$\min_{u,q,\mathbf{w},\mathbf{p}} \{\lambda \int_\Omega |q| + \frac{1}{2}\int (f-u)^2\}$$

$$\textit{with } q = \nabla \cdot \mathbf{n},\ \mathbf{n} = \frac{\mathbf{p}}{|\mathbf{p}|},\ \mathbf{p} = \langle \nabla u, 1\rangle.$$

$$\begin{aligned}
&E(u, q, \mathbf{p}, \mathbf{n}, \mathbf{m}, \lambda_1, \boldsymbol{\lambda_2}, \lambda_3, \boldsymbol{\lambda_4}) \\
&\qquad = \lambda \int_\Omega |q| + \frac{1}{2}\int (f-u)^2 + \frac{r_3}{2}\int (q - \nabla\cdot\mathbf{n})^2 \\
&\qquad + \int \lambda_3 (q - \nabla\cdot\mathbf{n}) + \frac{r_2}{2}\int |\mathbf{p} - \langle\nabla u, 1\rangle|^2 \\
&\qquad + \int \boldsymbol{\lambda_2}\cdot(\mathbf{p} - \langle\nabla u, 1\rangle) + \frac{r_1}{2}\int (|\mathbf{p}| - \mathbf{p}\cdot m)^2 \\
&\qquad + \int \boldsymbol{\lambda_1}\cdot(|\mathbf{p}| - \mathbf{p}\cdot m) + \frac{r_4}{2}\int |\mathbf{n} - \mathbf{m}|^2 \\
&\qquad + \int \boldsymbol{\lambda_4}\cdot(\mathbf{n} - \mathbf{m}) + \delta_{\mathbb{R}}(\mathbf{m}) \text{ (*–Indicator function–*)}
\end{aligned}$$